by O_1 and O_2. Then $\angle O_1A'O_2 = \angle O_1O'O_2 = \angle O_1O'A' + \angle O_2O'A' = \angle O'B'A' - \angle O'C'A' = \angle C'O'B' = \angle P'$. Thus $\angle A = \angle P'$. Similarly any angle in the figure $APBQCRO$ can be proved equal to the corresponding angle in the figure $P'A'Q'B'R'C'O'$.

III. The orthocentric line of either quadrilateral inverts into the circumcentric circle of the other.

For the pedal line, p, passes through the intersection of BCP and a line through O perpendicular to that line. Hence its inverse is a circle passing through the intersection of the circle $O'B'C'P'$ and a straight line through O' orthogonal to that circle. Hence the inverse of p is a circle passing through the opposite extremities of the diameters through O' of the circles $A'R'B'$, $B'C'P'$, $C'A'Q'$, $P'Q'R'$. Hence the inverse of the orthocentric line, o, which is parallel to p and twice as far from O,[6] is a circle touching the inverse of p at O' and with a radius half as large; that is, it is the circumcentric circle.[7]

IV. Many properties of a complete quadrilateral lead, on inversion, to new properties of this figure. However the results are often complicated and of little interest. One example will be given.

The theorem: "The circles on the three diagonals of the quadrilateral as diameters are coaxial, the radical axis being the orthocentric line,"[8] inverts into: "The three circles, (1) through A, P orthogonal to the circle OAP, (2) through B, Q orthogonal to OBQ, (3) through C, R orthogonal to OCR, are coaxial; and the circumcentric circle belongs to the same coaxial system."

A PROBLEM IN PROBABILITY.[1]

By C. S. JACKSON, R. M. Academy, Woolwich, England.

1. A problem first proposed by De Moivre and extended by Simpson was thrown into the following form by Laplace: If the numerical result of a single trial is equally likely to have any value between 0 and b, the chance that after n trials the sum of the results obtained shall be less than a is

$$\text{(i)} \qquad \frac{1}{b^n n!}\{a^n - n_1(a-b)^n + n_2(a-2b)^n \cdots\},$$

n_r denoting $n!/[r!(n-r)!]$ and the series being continued as long as $a - rb$ is positive.[2] In the following note an alternative mode of investigating (i) is used, which is intended to illustrate how each term of the formula arises.

2. Let $x_1 \cdots x_n$ be n positive items, each equally likely to have any value

---

[6] STEINER, *loc. cit.* DAVIES, *loc. cit.* CASEY, *loc. cit.*

[7] Cf. MCCLELLAND, *loc. cit.*

[8] DURELL, *Plane Geometry for Advanced Students*, Part I, Theorem 86, p. 188.

[1] The proof sheets of this article never reached us from the author, having probably been lost in ocean transit. EDITORS.

[2] See TODHUNTER, *History, etc., of Probability*, pp. 84, 208, 542.

between 0 and a, where a lies between the values rb and $(r+1)b$. The chance k_0 that their sum s is less than a is

$$a^{-n}\int_0^a dx_1\int_0^{a-\xi_1} dx_2\int_0^{a-\xi_2} dx_3\cdots\int_0^{a-\xi_{n-1}} dx_n,$$

where $\xi_r = x_1 + x_2 \cdots + x_r$.

This is a well-known integration, or may be worked out by putting

$$z_1 = a - \xi_1, \qquad z_2 = a - \xi_2, \qquad \cdots, \qquad z_n = a - \xi_n,$$

when it becomes

$$a^{-n}\int_0^a dx_1\int_0^{z_1} dz_2\cdots\int_0^{z_{n-1}} dz_n = a^{-n}\frac{a^n}{n!} = \frac{1}{n!}.$$

Again, the chance that $s < a$ and, at any rate, each of m specified items, say $x_1 \cdots x_m$, is greater than b, whatever the others may be, is

$$a^{-n}\int_b^{a-(m-1)b} dx_1\cdots\int_b^{a-(m-r-1)b-\xi_r} dx_{r+1}\cdots\int_b^{a-\xi_{m-1}} dx_m\cdots\int_0^{a-\xi_s} dx_{s+1}\cdots\int_0^{a-\xi_{n-1}} dx_n,$$

the limits being obtained by noticing that $x_1 > b$ and $x_1 < a-(m-1)b$, because at least $(m-1)b$ must be left to provide for $x_2 \cdots x_m$ each exceeding b. We may put for $\lambda > m$

$$z_\lambda = a - \xi_\lambda, \qquad \text{so that} \qquad \int_0^{a-\xi_{\lambda-1}} dx_\lambda = \int_0^{z_{\lambda-1}} dz_\lambda;$$

while for $\lambda \leqq m$ we put

$$z_\lambda = a-(m-\lambda)b - \xi_\lambda, \qquad \text{so that} \qquad \int_b^{a-(m-\lambda)b-\xi_{\lambda-1}} dx_\lambda = \int_0^{z_{\lambda-1}} dz_\lambda;$$

and then the integral becomes

$$a^{-n}\int_0^{a-mb} dz_1\int_0^{z_1} dz_2\cdots\int_0^{z_{n-1}} dz_n = \frac{(a-mb)^n}{a^n n!}.$$

3. The m specified items might be chosen in n_m ways, whence we would write

$$k_m = \frac{(a-mb)^n}{a^n m!(n-m)!},$$

and proceed to analyze k_m into components according to the exact number of items which exceed b.

If $u_0 =$ the chance that $s < a$, when *none* of the items $x_1 \cdots x_n$ exceeds b,
$u_1 =$ the chance that $s < a$, when *one* of the items $x_1 \cdots x_n$ exceeds b,
. . . .
$u_s =$ the chance that $s < a$, when *exactly* s of the items $x_1 \cdots x_n$ exceeds b,
and so on (the set ending with u_r, for it is impossible for more than r items to exceed b), then

(ii) $$k_m = u_m + (m+1)_m u_{m+1} \cdots + r_m u_r.$$

The cases which give rise to u_{m+l} are each counted $(m+l)_m$ times in k_m. Putting $m = 0, 1, \cdots r$ in turn in (ii) we obtain

$$
\begin{array}{llr}
k_0 = u_0 + u_1 + \cdots & + u_r. & (0) \\
k_1 = u_1 + 2_1 u_2 + \cdots & + r_1 u_r. & (1) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \\
k_m = & u_m + (m+1)_m u_{(m+1)} + \cdots + r_m u_r. & (m) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \\
k_{(m+l)} = & u_{(m+l)} + \cdots + r_{(m+l)} u_r. & (m+l) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \\
k_r = & r_r u_r. & (r)
\end{array}
$$

To solve these equations, multiply equation m by 1, $\cdots$, equation $(m+l)$ by $(-1)^l (m+l)_m$, $\cdots$, and add.

The coefficient of u_{m+l} in the result is

$$
(m+l)_m - (m+l)_{m+1}(m+1)_m + \cdots + (-1)^s (m+l)_{(m+s)}(m+s)_m \cdots + (-1)^l (m+l)_m,
$$

which is

$$
(m+l)_m \{1 - l_1 + \cdots + (-1)^s l_s \cdots + (-1)^l\} = (m+l)_m (1-1)^l = 0.
$$

Thus,

(iii) $\quad u_m = k_m - (m+1)_m k_{m+1} + (m+2)_m k_{m+2} \cdots + (-1)^{r-m} r_m k_m, \cdots$

and, in particular,

$$
u_0 = k_0 - k_1 + k_2 \cdots + (-1)^r k_r,
$$

where, as already shown,

$$
k_m = \frac{(a - mb)^n}{a^n m!(n-m)!}.
$$

4. Now the probability being u_0 that $s < a$ and that each of the items $x_1 \cdots x_n$ less than b, and the *a priori* probability being $(b/a)^n$ that $x_1 \cdots x_n$ shall each be less than b, then the probability that, when $x_1 \cdots x_n$ are each given less than b, their sum s shall be less than a is

$$
\left(\frac{a}{b}\right)^n \times u_0
$$

or

(iv) $$\frac{1}{b^n n!}\{a^n - n_1(a-b)^n + n_2(a-2b)^n \cdots\}$$

5. Again, from (iii),

$$u_m = \frac{(a - mb)^n}{a^n \cdot m!(n - m)!} - (m + 1)_m \frac{(a - mb - b)^n}{a^n(m + 1)!(n - m - 1)!} + \cdots$$
$$+ (-1)^s(m + s)_m \frac{(a - mb - sb)^n}{a^n(m + s)!(n - m - s)!} \cdots,$$

and the *a priori* probability being $n_m[(a - b)^m b^{n-m}/a^n]$ that exactly m items exceed b, the chance that, when m items exceed b, their sum s shall be less than a is

$$\text{(v)} \quad \frac{1}{n!(a - b)^m b^{n-m}} \{(a - mb)^n - (n - m)(a - mb - b)^n \cdots$$
$$+ (-1)^s(n - m)_s(a - mb - sb)^n \cdots\}.$$

This last result (v) is the chance that if, out of n positive items, m are equally likely to have any value between b and a, and the remainder to have any value less than b, then their sum shall be less than a.

6. The Hon. R. J. Strutt gave an interesting application of formula iv in the *Philosophical Magazine*, 6 series, Vol. 1, p. 311. The sum of the numerical departures from integral values of nine well-determined atomic weights is .809. If we suppose that an individual departure is equally likely to have any value between 0 and .5, the chance of the sum of nine departures being less than .809 proves to be .001159. The smallness of this value, he infers, gives some support to the well-known hypothesis that the atomic weights should be integers.

A SIMPLE GEOMETRICAL PARADOX.

PROPOSED BY J. L. COOLIDGE, Harvard University.

Suppose that we have an algebraic surface

$$x = \frac{f_1(u, v, w)}{f(u, v, w)}, \qquad y = \frac{f_2(u, v, w)}{f(u, v, w)}, \qquad z = \frac{f_3(u, v, w)}{f(u, v, w)},$$
$$F(u, v, w) = 0.$$

We shall assume that this surface has no singular curve, an assumption which still leaves us in what we may call the *general* case, since the discriminant of a polynomial in three variables does not vanish identically. Let us cut this surface by an arbitrary plane which does not pass through any isolated singularity which the surface may possess, a *general* plane we might say. The coördinates of the points of the curve of intersection are algebraic functions of a single parameter, and the same is true of the sine of the angle which the given plane makes with the tangent plane to the surface at the points of the curve.

Suppose first, that this algebraic function is not a constant. It must, then,